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Fuzzy Events and Fuzzy Logics in Classical Information Systems

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Classical information systems are introduced in the framework of measure and integration theory. The measurable characteristic functions are identified with the exact events while the fuzzy events are the real measurable functions whose range is contained in the unit interval. Two orthogonality relations are introduced on fuzzy events, the first linked to the fuzzy logic and the second to the fuzzy structure of partial a Baer*-ring. The fuzzy logic is then compared with the "empirical" fuzzy logic induced by the classical information system. In this context, quantum logics could be considered as those empirical fuzzy logics in which it is not possible to have preparation procedures which provide physical systems whose "microstate" is always exactly defined.

INTRODUCTION

At present fuzzy theories are assuming importance in a wide variety of scientific branches such as "information theory," "pattern recognitions," "theoretical physics," and so on.

As to the last, the kind of "fuzziness" which interests us, regarding both classical and quantum physics, concerns the fact that the actually realizable experimental apparatuses, which are concretely realizable in practice, suffer from a not eliminable noise, suggesting strongly that these devices are "fuzzy." According to Giles [9], "The phsical significant assertions of classical or quantum mechanics must refer to physical or concretely realizable devices, for no other devices can actually be realized. Thus the concept of a [fuzzy] device must appear in the theory. It is a serious fault of conventional quantum mechanics (and, for that matter, of classical mechanics) that this concept is not present." Usually it is assumed that, at least in principle, it is always possible to get such technological refinements in experimental apparatuses to yield measures of a single observable as being produced by "exact" devices. In this way a theory describing this practical situation can be based upon "exact algorithms" without appreciable damage and consequently the corresponding fuzzy "machinery" could be neglected. This is not right at all, for a fuzzy mathematical theory gives useful information about the underlying exact structure and, at any rate, it is often interesting to see, as "exact" mathematical objects can be approximated, for instance, by sequences of "fuzzy" objects of the same kind.

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Since a physical theory consists of a mathematical structure together with a set of rules of interpretation, these last results al so give us interesting information about the feature of realistic, i.e., imperfectly accurate, devices which provide sharper and sharper approximation of exact, i.e., idealized, devices.

In this work we deal solely with the fuzziness involving a classical information system, that is, a mathematical structure which could provide the formal scheme, for instance, of classical probability theory or, also, of classical statistical mechanics of a well-singled-out physical situation. In particular, the exact events turn out to be the measurable subsets of a measurable space and they are identified with the corresponding characteristic functionals. The order properties of the exact events are studied in Section 2, emphasizing those features which could successively be generalized to the fuzzy case. Fuzzy events are then introduced in Section 3 according to the Zadeh approach to these questions, and then their intrinsic ordered, or, more generally, algebraic, structures are taken into account. In this way, we agree with the De Luca and Termini program of developing the "algebraic analysis" of fuzzy set theory presented in [3] and carried on in [4].

In particular, we stress that the set of fuzzy events is a distributive σ-lattice with two possible weak orthocomplementations, one of which is the generalization of the usual orthocomplementation defined on exact events and the other the Brouwerian orthocomplementation. Introducing the set of complex fuzzy events and extending in a natural way the traditional definition of fuzzy event, it is shown that they form a Baer*-semigroup according to Foulis (see [5, 6]), which is also a partial Baer*-ring relative to a sum operation defined only for couples of mutually orthogonal elements.

This last result is very interesting since it is obtainable in more general situations, such as the Hilbert space model or as the generalized probability (or quantum logic) approach to (nonrelativistic) quantum mechanics. In fact, we shall check in other works of ours that in these cases we can yet introduce the set of complex fuzzy events with an algebraic structure of partial Baer*-ring.

Moreover, the corresponding fuzzy events in the Hilbert-space approach to quantum mechanics enclose the fuzzy events considered by Prugovecki (see [15]) and the localization of fuzzy events introduced in [17, 18]. These Hilbert fuzzy events are strictly linked to the operations of filtering processes of the operational approach to quantum probability introduced by Davies and Lewis in [2], in agreement also with the Pool axiomatic approach presented in [14].

1. Classical Information Systems

A classical information system on a nonempty set Γ is a structure $(\mathscr{E}(\Gamma), \mathscr{N}(\Gamma), \mathscr{B}(\Gamma), P)$ which we shall proceed to introduce axiomatically in the present section. In the first place, it must satisfy the following:

AXIOM 1. *$\mathscr{E}(\Gamma)$ is a σ-algebra of sets on Γ, i.e., $\mathscr{E}(\Gamma)$ is a family of subsets of Γ such that*

(1.i) $\phi \in \mathscr{E}(\Gamma)$;

(1.ii) *if $E \in \mathscr{E}(\Gamma)$ then $E^C \in \mathscr{E}(\Gamma)$;*

(1.iii) *if $\{E_n : n \in \mathbb{N}\}$ is a countable set whose elements belong to $\mathscr{E}(\Gamma)$ then $\cup E_n$ also belongs to $\mathscr{E}(\Gamma)$.*

It is a trivial consequence of this axiom that $\Gamma \in \mathscr{E}(\Gamma)$ and $\cap E_n \in \mathscr{E}(\Gamma)$ also.

A mapping $f: \Gamma \to \mathbb{C}$ is said to be *measurable* iff $f^{-1}(\Delta) \in \mathscr{E}(\Gamma)$ for every $\Delta \in \mathscr{B}(\mathbb{C})$, where $\mathscr{B}(\mathbb{C})$ is the natural Borel σ-algebra on $\mathbb{C}$.

Of course, if $f_n: \Gamma \to \mathbb{C}$, $f: \Gamma \to \mathbb{C}$, and $g: \Gamma \to \mathbb{C}$ are measurable mappings then the following mappings also are measurable:

(1) $(f+g)(x) := f(x) + g(x)$;

(2) $(\lambda f)(x) := \lambda f(x)$;

(3) $(f \cdot g)(x) := f(x) \cdot g(x)$;

(4) $\sup f_n := \sup\{f_n(x) : x \in \Gamma\}$;

(5) $\inf f_n ;= \inf\{f_n(x) : x \in \Gamma\}$;

(6) $\bar{f}(x) := \overline{f(x)}$;

(7) $|f|(x) := |f(x)|$.

Let A be a subset of Γ; the *characteristic functional* $\chi_A: \Gamma \to \mathbb{C}$ of A is defined by the law

$$\begin{aligned} \chi_A(x) &:= 0, \qquad x \notin A \\ &:= 1, \qquad x \in A, \end{aligned}$$

and χ_A is measurable iff A is an element of $\mathscr{E}(\Gamma)$.

The *spectrum* of a measurable mapping $f: \Gamma \to \mathbb{C}$ is the Borel subset of $\mathbb{C}$ obtained by the closure, with respect to the natural topology of $\mathbb{C}$, of the range of f:

$$\sigma(f) := \operatorname{cl} \operatorname{ran}(f) = \operatorname{cl}\{f(x) \in \mathbb{C} : x \in \Gamma\}.$$

The *resolvent* of f is the open subset of $\mathbb{C}$:

$$\rho(f) := \mathbb{C} \backslash \sigma(f).$$

A measurable mapping is said to be *bounded* iff $\sigma(f)$ is a bounded (closed) subset of $\mathbb{C}$, while it is said to be *real* iff $\sigma(f) \subseteq \mathbb{R}$. For instance, for every $E \in \mathscr{E}(\Gamma)$, the characteristic functional χ_E is a real bounded measurable mapping for which $\sigma(\chi_E) \subseteq \{0, 1\}$.

Let $f: \Gamma \to \mathbb{C}$ be a measurable mapping and $u: \mathbb{C} \to \mathbb{C}$ a Borel function; then $(u \circ f): \Gamma \to \mathbb{C}$, defined by $(u \circ f)(x) := u(f(x))$ is a measurable mapping.

AXIOM 2. *$\mathcal{N}(\Gamma)$ is a set of measurable mappings such that:*

(2.i) $\chi_E \in \mathcal{N}(\Gamma)$ *for every* $E \in \mathcal{E}(\Gamma)$;

(2.ii) *if* $f \in \mathcal{N}(\Gamma)$ *then* $(u \circ f) \in \mathcal{N}(\Gamma)$ *for every Borel function* u;

(2.iii) *if* f *and* g *are in* $\mathcal{N}(\Gamma)$ *and* λ *is any complex number, then* $f + g$, λf $f \cdot g$, $\bar{f}$, *and* $|f|$ *are also in* $\mathcal{N}(\Gamma)$;

(2.iv) *if* $\{f_n : n \in \mathbb{N}\}$ *is a sequence of elements of* $\mathcal{N}(\Gamma)$ *then* $\sup(f_n)$ *and* $\inf(f_n)$ *belong to* $\mathcal{N}(\Gamma)$.

We shall denote with $\mathcal{O}(\Gamma)$ the set of all the real elements of $\mathcal{N}(\Gamma)$.

A finite measure on $\mathcal{E}(\Gamma)$ is a mapping $\mu: \mathcal{E}(\Gamma) \to \mathbb{R}_+$ such that

(fm.i) $\mu(\Gamma) < +\infty$;

(fm.ii) $\mu(\cup E_n) = \sum \mu(E_n)$ for every countable family of mutually disjoint elements on $\mathcal{E}(\Gamma)$.

From this definition we get that a finite measure also satisfies the conditions

(8) $E_1 \subseteq E_2$ implies $\mu(E_1) \leqslant \mu(E_2)$;

(9) $\mu(E) + \mu(E^C) = \mu(\Gamma)$;

(10) $\mu(\varnothing) = 0$.

Moreover, if f is a bounded measurable function, then for every $\mu \in \mathcal{B}(\Gamma)$ there exists the integral

(11) $\mu(f) := \int_\Gamma f \, d\mu$,

and this quantity is real if f is real. If f is a measurable function, we generalize notation (11), denoting also with $\mu(f)$ the integral $\int_\Gamma f \, d\mu$, if it exists.

AXIOM 3. *$\mathcal{B}(\Gamma)$ is a nonempty set of finite measures with the following properties:*

(3.i) (Completeness property). *For every* $E \in \mathcal{E}(\Gamma)$ *these exists* $\mu_E \in \mathcal{B}(\Gamma)$ *such that* $\mu_E(E) = \mu_E(\Gamma)$;

(3.ii) (distinguishing property). *Let* $f_1, f_2 \in \mathcal{N}(\Gamma)$ *with* f_1 *bounded, then* $\mu(f_1) = \mu(f_2)$ *for every* $\mu \in \mathcal{B}(\Gamma)$*-implies* $f_1(x) = f_2(x)$ *for every* $x \in \Gamma$;

(3.iii) (order determining property). *If* $E_1, E_2 \in \mathcal{E}(\Gamma)$ *then* $\mu(E_1) \leqslant \mu(E_2)$ *for every* $\mu \in \mathcal{B}(\Gamma)$ *implies* $E_1 \subseteq E_2$.

A classical information system is said to have the mixing property iff for the set $\mathcal{B}(\Gamma)$ *the additional condition*

(3.iv) $\mu_1 + \mu_2 \in \mathcal{B}(\Gamma)$ *for every* μ_1 *and* μ_2 *belonging to* $\mathcal{B}(\Gamma)$; $\lambda\mu \in \mathcal{B}(\Gamma)$ *for every* $\lambda \in \mathbb{R}_+$ *and every* $\mu \in \mathcal{B}(\Gamma)$

holds.

From a physical point of view the previous axioms can be used to describe a quite usual experimental situation regarding a certain physical system, which can be prepared under well-defined and repeatable conditions and on which can be performed certain yes–no observations, i.e., observations giving only two possible answers (either yes or no) as a result of their measure. The elements of $\mathscr{E}(\Gamma)$ represent the ***elementary*** (***exact***) ***events*** of this physical situation and any element of $\mathscr{E}(\Gamma)$ is an (*exact*) *event* or an (*exact*) *yes–no experiment*. The elements of $\mathscr{B}(\Gamma)$ describe the concretely realizable ***preparation procedures*** of the system under examination. Therefore, a couple $(\mu, E) \in \mathscr{B}(\Gamma) \times \mathscr{E}(\Gamma)$ represents an ***elementary experiment*** consisting of a ***preparation part*** μ and a yes–no ***observation part*** E. The real quantity $\mu(E)$ is the ***intensity*** or ***frequence*** of yes occurence of the event E when the system is prepared according to μ.

preparation part: μ	$\rightsquigarrow$	observation part: E

$$P(\mu, E) = \frac{\mu(E)}{\mu(\Gamma)} .$$

Under these conditions, the mapping

$$P\colon \mathscr{B}(\Gamma) \times \mathscr{E}(\Gamma) \to [0, 1], \qquad (\mu, E) \to P(\mu, E) := \frac{\mu(E)}{\mu(\Gamma)} \tag{12}$$

is the probability of occurrence of E if the preparation of the system is made by μ.

The various ***observable*** quantities, which can be concretely measured on the system, are represented by the elements of $\mathscr{N}(\Gamma)$ which are real measurable mappings, i.e., by the elements of $\mathscr{O}(\Gamma)$.

The quantity

$$\langle \mu; f \rangle := \frac{\mu(f)}{\mu(\Gamma)}, \tag{13}$$

if it exists, is interpreted as the *expectation* or *average value* of the observable f when the system is prepared by the procedure μ.

Of course, $\mu(\chi_E) = \int_\Gamma \chi_E \, d\mu = \mu(E)$, and therefore we get that

$$\langle \mu; \chi_E \rangle = P(\mu, E) \in [0, 1], \tag{14}$$

and in the following we shall also use the notation

$$P(\mu, \chi_E) := \langle \mu; \chi_E \rangle = P(\mu, E). \tag{15}$$

2. Order Properties of Exact Events: Orthogonal Projections

In this section we shall consider the properties of a classical information system from the point of view of a partial order, which can be introduced on the family of exact events. The results obtained will allow us to introduce the set of fuzzy events as a natural generalization of the notion of an exact event.

It is not difficult to prove the following:

Proposition 2.1. *Let Γ be a nonempty set and let $\mathscr{E}(\Gamma)$ be a σ-algebra of subset of Γ; then*

$$(\mathscr{E}(\Gamma),\ \subseteq,\ \varnothing,\ \Gamma,\ \cdot^C),$$

where $\subseteq$ is the usual inclusion relation and $\cdot^C$, the complementation mapping on sets, is a boolean σ-lattice with least element $\varnothing$ and geatest element Γ.

More exactly, $\subseteq$ is a partial order (poset, for short) relation on $\mathscr{E}(\Gamma)$, i.e., it satisfies the conditions:

(Or.i) $E \subseteq E$ for every $E \in \mathscr{E}(\Gamma)$;
(Or.ii) $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ imply $E_1 = E_2$;
(Or.iii) $E_1 \subseteq E_2$ and $E_2 \subseteq E_3$ imply $E_1 \subseteq E_3$.

The poset $\mathscr{E}(\Gamma)$ is bounded because there exist $\varnothing$ and $\Gamma \in \mathscr{E}(\Gamma)$ such that

$$\varnothing \subseteq E \subseteq \Gamma \qquad \text{for every } E \in \mathscr{E}(\Gamma).$$

Moreover for every countable family $\{E_n : n \in \mathbb{N}\}$ of elements of $\mathscr{E}(\Gamma)$ there exist the lub and the glb with regard to the order relation of set inclusion, i.e., the following conditions are satisfied:

(σl.i) $\text{lub}\{E_n : n \in \mathbb{N}\} = \cup\{E_n : n \in \mathbb{N}\}$;
(σl.ii) $\text{glb}\{E_n : n \in \mathbb{N}\} = \cap\{E_n : n \in \mathbb{N}\}$.

In this manner $\mathscr{E}(\Gamma)$ is a σ-lattice.

The complementation mapping $\cdot^C : \mathscr{E}(\Gamma) \to \mathscr{E}(\Gamma)$, which assigns to each element E of $\mathscr{E}(\Gamma)$ the set $E^C = \Gamma \backslash E \in \mathscr{E}(\Gamma)$, is a (strong, nondegenerate) orthocomplementation [1], because the following obvious properties hold:

(Oc.i) $(E^C)^C = E$ for every $E \in \mathscr{E}(\Gamma)$;
(Oc.ii) $E_1 \subseteq E_2$ implies $E_2{}^C \subseteq E_1{}^C$;
(Oc.iii) $E \cup E^C = \Gamma$ for every $E \in \mathscr{E}(\Gamma)$.

At last, the following mutually equivalent distributive properties are easy to prove:

(d.i) $E_1 \cap (E_2 \cup E_3) = (E_1 \cap E_2) \cup (E_1 \cap E_3)$ for every $E_1, E_2, E_3 \in \mathscr{E}(\Gamma)$;
(d.ii) $E_1 \cup (E_2 \cap E_3) = (E_1 \cup E_2) \cap (E_1 \cup E_3)$ for every $E_1, E_3, E_1 \in \mathscr{E}(\Gamma)$.

Two exact events E_1 and E_2 are said to be *orthogonal* or *mutually disjoint*, written $E_1 \perp E_2$, iff $E_1 \subseteq E_2{}^C$ (or equivalently, $E_2 \subseteq E_1{}^C$ or $E_1 \cap E_2 = \varnothing$); $\perp$ is a (strong and nondegenerate) orthogonality relation on $\mathscr{E}(\Gamma)$ [1].

The set $\mathscr{E}(\Gamma)$ of exact events can be considered a poset with regard to an order relation quite different from the one introduced in Proposition 2.1. Precisely, if we define the binary relation

$$E_1 \leqslant E_2 \quad \text{iff} \quad \mu(E_1) \leqslant \mu(E_2) \qquad \text{for every } \mu \in \mathscr{B}(\Gamma) \tag{1}$$

then $\leqslant$ is obviously reflexive and transitive and, by (3.iii), is also antisymmetric. The poset $(\mathscr{E}(\Gamma), \leqslant)$ is bounded by the least element $\varnothing$: $\varnothing \leqslant E$ for every $E \in \mathscr{E}(\Gamma)$, and by the greatest element Γ: $E \leqslant \Gamma$ for every $E \in \mathscr{E}(\Gamma)$. In fact

$$0 = \mu(\varnothing) \leqslant \mu(E) \leqslant \mu(\Gamma) \qquad \text{for every } \mu \in \mathscr{B}(\Gamma). \tag{2}$$

PROPOSITION 2.2. *The mapping*

$$\mathscr{E}(\Gamma) \to \mathscr{E}(\Gamma), \qquad E \to E' := \Gamma \backslash E = E^C \tag{3}$$

is an orthocomplementation on $(\mathscr{E}(\Gamma), \leqslant, \varnothing, \Gamma)$. *Indeed, the following properties hold:*

$$(E')' = E \qquad \textit{for every } E \in \mathscr{E}(\Gamma); \tag{4}$$

$$E_1 \leqslant E_2 \qquad \textit{implies} \qquad E_2' \leqslant E_1', \tag{5}$$

and denoted with $E_1 \vee E_2$ *and* $E_1 \wedge E_2$, *respectively, the* lub *and the* glb *of* $\{E_1, E_2\}$, *if they exist, with regard to the order relation* (1), *give us*

$$E \vee E' = \Gamma \qquad \textit{for every } E \in \mathscr{E}(\Gamma). \tag{6}$$

Proof. We shall prove (5) and (6).

Let $E_1 \leqslant E_2$; then $\mu(E_1) \leqslant \mu(E_2)$ for every $\mu \in \mathscr{B}(\Gamma)$. But $0 \leqslant \mu(E_1) + \mu(E_1') = \mu(\Gamma)$ and $\mu(E_2) + \mu(E_2') = \mu(\Gamma)$ imply $0 \leqslant \mu(E_1) + \mu(E_1') = \mu(E_2) + \mu(E_2')$ for every $\mu \in \mathscr{B}(\Gamma)$. Therefore $\mu(E_2') \leqslant \mu(E_1')$ for every $\mu \in \mathscr{B}(\Gamma)$, i.e., $E_2' \leqslant E_1'$.

Let us consider $\{E, E'\}$ and let $\hat{E}$ be an upper bound of $\{E, E'\}$, i.e., $E \leqslant \hat{E} \leqslant \Gamma$ and $E' \leqslant \hat{E} \leqslant \Gamma$. Then we get $\mu(E) \leqslant \mu(\hat{E}) \leqslant \mu(\Gamma)$ and $\mu(E') \leqslant \mu(\hat{E}) \leqslant \mu(\Gamma)$ for every $\mu \in \mathscr{B}(\Gamma)$. From this result and from (3.iii) it follows that $E \subseteq \hat{E} \subseteq \Gamma$ and $E^C \subseteq \hat{E} \subseteq \Gamma$, and then

$$\Gamma = E \cup E^C \subseteq \hat{E} \subseteq \Gamma,$$

from which we get $\hat{E} = \Gamma$. Therefore $E \vee E' = \Gamma$.

The orthocomplemented poset $(\mathscr{E}(\Gamma), \leqslant, \varnothing, \Gamma, \cdot')$ is obviously isomorphic to the Boolean σ-lattice $(\mathscr{E}(\Gamma), \subseteq, \varnothing, \Gamma, \cdot^C)$ because (3.iii) and (8), Section 1, imply

$$E_1 \subseteq E_2 \qquad \text{iff} \qquad E_1 \leqslant E_2 \tag{7}$$

and, moreover,

$$E^C = E'. \tag{8}$$

Therefore, $(\mathscr{E}(\Gamma), \leqslant, \varnothing, \Gamma, \cdot')$ is also a Boolean σ-lattice for which

$$\bigvee\{E_n: n \in \mathbb{N}\} = \bigcup\{E_n: n \in \mathbb{N}\}$$
$$\bigwedge\{E_n: n \in \mathbb{N}\} = \bigcap\{E_n: n \in \mathbb{N}\}.$$

The orthogonality relation induced by the orthocomplementation (3) is now expressed by the proposition

$$E_1 \perp E_2 \quad \text{iff} \quad 0 \leqslant \mu(E_1) + \mu(E_2) \leqslant \mu(\Gamma) \qquad \text{for every } \mu \in \mathscr{B}(\Gamma). \tag{9}$$

With the aim of introducing fuzzy events, we consider now a Boolean σ-algebra, whose elements are the characteristic mappings of all the exact events. A measurable function is said to be an *orthogonal projection* iff $f^2 = f$. From this definition we get

PROPOSITION 2.3. *Let $f: \Gamma \to \mathbb{C}$ be a measurable mapping; then the following propositions are equivalent:*

(i) *f is a projection;*

(ii) $\sigma(f) \subseteq \{0, 1\}$;

(iii) *there exists $E \in \mathscr{E}(\Gamma)$ such that $f = \chi_E$.*

In the following, the set of all projections is denoted by $\Pi(\Gamma)$, i.e., $\Pi(\Gamma) = \{\chi_E: E \in \mathscr{E}(\Gamma)\}$. From the previous proposition and Axiom 2 it is obvious that

$$\Pi(\Gamma) \subseteq \mathscr{O}(\Gamma). \tag{10}$$

Moreover, it is easy to prove that

$$\chi: \mathscr{E}(\Gamma) \to \Pi(\Gamma), \qquad E \to \chi_E \tag{11}$$

is a one-to-one mapping from $\mathscr{E}(\Gamma)$ onto $\Pi(\Gamma)$.

Setting for short $\mathbf{0} := \chi_\phi$ and $\mathbf{1} := \chi_\Gamma$, the set of projection observables can be considered an ordered structure $(\Pi(\Gamma), \subseteq, \mathbf{0}, \mathbf{1}, ')$ if we introduce the order relation

$$\chi_{E_1} \subseteq \chi_{E_2} \qquad \text{iff} \qquad \chi_{E_1}(x) \leqslant \chi_{E_2}(x) \qquad \text{for every } x \in \Gamma \tag{12}$$

and if we define

$$(\chi_E)' = \mathbf{1} - \chi_E. \tag{13}$$

In this manner $\Pi(\Gamma)$ turns out to be a poset bounded by $\mathbf{0}$ and $\mathbf{1}$ with orthocomplementation. From this orthocomplementation we induce (see [1]) the othogonality relation:

$$\chi_{E_1} \perp \chi_{E_2} \qquad \text{iff} \qquad \chi_{E_1} \subseteq (\chi_{E_2})'. \tag{14}$$

Trivially, conditions (12) and (14) can be equivalently written in the following way:

$$\chi_{E_1} \subseteq \chi_{E_2} \qquad \text{iff} \qquad \chi_{E_1} = \chi_{E_1} \cdot \chi_{E_2}; \tag{12a}$$

$$\chi_{E_1} \perp \chi_{E_2} \qquad \text{iff} \qquad 0 \leqslant \chi_{E_1}(x) + \chi_{E_2}(x) \leqslant 1 \qquad \text{for every } x \in \Gamma; \tag{14a}$$

or

$$\chi_{E_1} \perp \chi_{E_2} \qquad \text{iff} \qquad \mathbf{0} = \chi_{E_1} \cdot \chi_{E_2}. \tag{14b}$$

This being stated, it is easy to prove the following result, which shows that the mapping introduced by (11) is an isomorphism between Boolean σ-lattices:

PROPOSITION 2.4. *Let* χ_{E_1} *and* $\chi_{E_2} \in \Pi(\Gamma)$; *then*

(i) $E_1 \subseteq E_2$ *iff* $\chi_{E_1} \subseteq \chi_{E_2}$;

(ii) $\chi_{E^C} = (\chi_E)'$;

(iii) $E_1 \perp E_2$ *iff* $\chi_{E_1} \perp \chi_{E_2}$.

The behavior of $\Pi(\Gamma)$, in relation to algebraic operations on functions, is given by the following properties:

$$\chi_{E_1} + \chi_{E_2} \in \Pi(\Gamma) \qquad \text{iff} \qquad \chi_{E_1} \perp \chi_{E_2}; \tag{15}$$

$$\chi_{E_1} \cdot \chi_{E_2} \in \Pi(\Gamma) \qquad \text{for every } \chi_{E_1}, \chi_{E_2} \in \Pi(\Gamma); \tag{16}$$

let $E \neq \varnothing$; then

$$\lambda \chi_E \in \Pi(\Gamma) \qquad \text{iff} \qquad \lambda \text{ is } 0 \text{ or } 1. \tag{17}$$

In particular, if $\chi_E \in \Pi(\Gamma)$ with $E \supseteq \varnothing$ then $-\chi_E \notin \Pi(\Gamma)$.

With regard to the order relation (12) we have that

$$\chi_{E_1} \cap \chi_{E_2} = \chi_{E_1} \cdot \chi_{E_2} = \inf\{\chi_{E_1}(x), \chi_{E_2}(x): x \in \Gamma\}; \tag{18}$$

$$\chi_{E_1} \cup \chi_{E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1} \cdot \chi_{E_2} = \sup\{\chi_{E_1}(x), \chi_{E_2}(x): x \in \Gamma\}. \tag{19}$$

We can also consider a structure $(\Pi(\Gamma), \leqslant, \mathbf{0}, \mathbf{1}, ')$, where the order relation is defined by

$$\chi_{E_1} \leqslant \chi_{E_2} \qquad \text{iff} \qquad \mu(\chi_{E_1}) \leqslant \mu(\chi_{E_2}) \qquad \text{for every } \mu \in \mathscr{B}(\Gamma) \tag{20}$$

and the orthocomplemented element associated to χ_E is $(\chi_E)' = \mathbf{1} - \chi_E$; but the axioms of a classical information system assure us that this structure is isomorphic to those we have seen in the present section.

At any rate, in conclusion, the previous results allow us to identitfy $\mathscr{E}(\Gamma)$ with $\Pi(\Gamma)$ as a Boolean σ-lattice, considering E and χ_E as the same element. Furthermore, this is physically justified by the fact that the occurrence probabilities $P(\mu, E) = P(\mu, \chi_E)$ are the same for every preparation procedure $\mu \in \mathscr{B}(\Gamma)$, and then the event E and the projection observable χ_E are not distinguishable by experiment built up inside our classical information system.

3. Fuzzy Events

From the consideration of the previous section it is natural to define a *fuzzy event* as any observables $F \in \mathscr{O}(\Gamma)$ such that $\sigma(F) \subseteq [0, 1]$. Equivalently, a fuzzy event is a measurable mapping $F: \Gamma \to [0, 1]$ belonging to $\mathscr{N}(\Gamma)$. From this definition we get that

$$0 \leqslant \mu(F) \leqslant \mu(\Gamma) = \mu(\mathbf{1})$$

and then

$$0 \leqslant \frac{\mu(F)}{\mu(\mathbf{1})} \leqslant 1 \qquad \text{for every } \mu \in \mathscr{B}(\Gamma).$$

Therefore, denoting by $\mathscr{F}(\Gamma)$ the set of all fuzzy events, the exact events are also fuzzy events:

$$\Pi(\Gamma) \subseteq \mathscr{F}(\Gamma), \tag{1}$$

and we can introduce the mapping

$$P: \mathscr{B}(\Gamma) \times \mathscr{F}(\Gamma) \to [0, 1]$$

defined by the law

$$P(\mu; F) := \frac{\mu(F)}{\mu(\mathbf{1})} = (\mu : F), \tag{2}$$

which can be interpreted as the *probability of occurrence of the fuzzy event* F when the system is prepared according to the procedure μ.

The identification between the Boolean σ-lattices

$$\mathscr{E}(\Gamma) \equiv \Pi(\Gamma), \qquad E \equiv \chi_E$$

shows that the restriction to $\mathscr{B}(\Gamma) \times \mathscr{E}(\Gamma)$ of the mapping P defined by (2) is exactly the mapping P defined by (15), Section 1:

$$P(\mu; \chi_E) = P(\mu; E).$$

Paraphrasing the Zadeh definition of a fuzzy set [22], we can say that a *fuzzy event* is an "event" with a continuum of grades of membership. Indeed "a *fuzzy event* F in Γ is characterized by a *membership measurable function* $F(x)$, which associates with each point in Γ a real number in the interval [0, 1], with the value of $F(x)$ at x representing the grade of membership of x in F."

In this mammer, "(···) the notion of a fuzzy [event is] a convenient point of departure for the construction of a conceptual framework ··· [which] ··· provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership (···)". These considerations agree with the remark that "more often than not, the classes of objects encountered in the real pysical worl do not have precisely defined criteria of membership. (···) Nevertheless, the fact remains that such imprecisely defined classes play an important role in human thinking."

Of course, an exact event is characterized by a membership measurable functional which can take on only the two values 0 and 1, i.e., by the characteristic functional of a certain element in $\mathscr{E}(\Gamma)$.

The notion of "belonging," which plays a fundamental role in the case of ordinary [events], does not have the same role in the case of fuzzy [events]. Thus it is not meaningful to speak of a point x "belonging" to a fuzzy event F except in the trivial sense of $F(x)$ being positive [i.e., $x \in F^{-1}((0, 1])$].

More exactly, one can say that "x belongs to F" iff $F(x) = 1$; "x does not belongs to F" iff $F(x) = 0$; and "x has an indeterminate status relative to F" iff $0 < F(x) < 1$. Introducing the three truth values: $T(F(x) = 1)$, $F(F(x) = 0)$, and $U(0 < F(x) < 1)$, this leads to a three-valued logic.

According to these considerations we can assign to each fuzzy event F two exact events

$$E_m(F) ;= F^{-1}(\{1\})$$
$$E_M(F) := F^{-1}((0, 1])$$

called, respectively, the *least and the greatest exact events associated* to F.

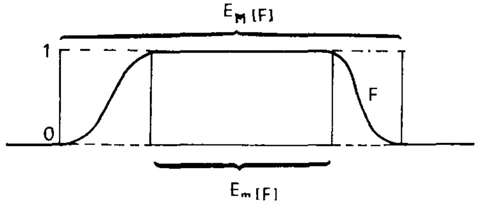

FIGURE 1

Figure 1 represents an exemplification of these concepts. A couple $(E, F) \in \mathscr{E}(\Gamma) \times \mathscr{F}(\Gamma)$ is said to be an *associated* couple of exact–fuzzy events iff

$$E_m(F) \subseteq E \subseteq E_M(F). \tag{3}$$

From the distinguishing property of bounded observables (3.ii), we have that, for two fuzzy events,

$$F_1 = F_2 \quad \text{iff} \quad \mu(F_1) = \mu(F_2) \qquad \text{for every } \mu \in \mathscr{B}(\Gamma), \tag{4}$$

and therefore we can introduce the order relation on $\mathscr{F}(\Gamma)$ defined by

$$F_1 \leqslant F_2 \quad \text{iff} \quad \mu(F_1) \leqslant \mu(F_2) \qquad \text{for every } \mu \in \mathscr{B}(\Gamma), \tag{5}$$

which is the extension of (20), Section 2.

We remark that, while $F_1 = F_2$ in (4) means that $F_1(x) = F_2(x)$ for every $x \in \Gamma$, in general, the relation $F_1 \leqslant F_2$ in (5) does not imply $F_1(x) \leqslant F_2(x)$ for every $x \in \Gamma$. In this way the order relation

$$F_1 \leqslant F_2 \quad \text{iff} \quad \mu(F_1) \leqslant \mu(F_2) \qquad \text{for every } \mu \in \mathscr{B}(\Gamma), \tag{5}$$

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We remark that, while $F_1 = F_2$ in (4) means that $F_1(x) = F_2(x)$ for every $x \in \Gamma$, in general, the relation $F_1 \leqslant F_2$ in (5) does not imply $F_1(x) \leqslant F_2(x)$ for every $x \in \Gamma$. In this way the order relation

$$F_1 \subseteq F_2 \quad \text{iff} \quad F_1(x) \leqslant F_2(x) \qquad \text{for every } x \in \Gamma, \tag{6}$$

which has been introduced by Zadeh [22] is different from (5) and, evidently, the following relations hold between them:

$$F_1 \subseteq F_2 \qquad \text{implies} \qquad F_1 \leqslant F_2\,, \qquad F_1 \quad \text{and} \quad F_2 \in \mathscr{F}(\Gamma); \tag{7}$$

$$E_1 \subseteq E_2 \qquad \text{iff} \qquad E_1 \leqslant E_2\,, \qquad E_1 \quad \text{and} \quad E_2 \in \mathscr{E}(\Gamma), \tag{8}$$

where $E_1 \subseteq E_2$ stands for $\chi_{E_1} \subseteq \chi_{E_2}$.

Generalizing the conclusions of Section 1, a couple $(\mu, F) \in \mathscr{B}(\Gamma) \times \mathscr{F}(\Gamma)$ represents an actually realizable elementary experiment consisting of a *preparation part* μ and a *yes–no fuzzy observation part* F. The probability of occurrence of the fuzzy event F, if the physical system is prepared according to μ, is expressed by

$$P(\mu, F) = \frac{\mu(F)}{\mu(1)} = \frac{\int_\Gamma F\, d\mu}{\int_\Gamma 1\, d\mu}\,.$$

Like exact events, which can be considered exact yes–no observables, fuzzy events are yes–no observables for which the experimental noise present in any *concretely realizable* apparatus is taken into account.

Quoting Giles [9], "It is clear that no *actual* device behaves [exactly]. Rather [$\cdots$] a *real* spectrometer is characterized by a probability of response which is at least a continuous function [$\cdots$]; for near the ends of its range the signal becomes dominated by noise, so that the probability of response drops continuously from 1 to 0."

This continuous underlining, that fuzzy events represent real, actual, concretely realizable experiments, means that we are dealing with a physical theory which describes the realistic feature in the physical inquire of nature.

Among other things, this is the reason for which we have not assumed that *all* the finite measures represent actually realizable physical preparation procedures and that *all* the measurable functions represent actually realizable observation procedures. Is is the analysis of each particular physical situation which allows us to set up a one-to-one correspondence between a certain class of actually realizable preparation (observation) devices and a corresponding element of $\mathscr{B}(\Gamma)$ ($\mathscr{N}(\Gamma)$).

4. Classical Fuzzy Logic of Degenerate Type

Concerning the order (6) the poset $(\mathscr{F}(\Gamma), \subseteq, 0, 1)$ is a *distributive σ-lattice with degenerate orthocomplementation.*

In fact, denoting with $F_1 \cup F_2$ the lub of $\{F_1, F_2\}$ and with $F_1 \cap F_2$ the glb of $\{F_1, F_2\}$, we now shall show:

Proposition 4.1.

$$F_1 \cup F_2 \quad \textit{exists in } (\mathscr{F}(\Gamma), \subseteq) \qquad \textit{and} \qquad F_1 \cup F_2 = \sup\{F_1, F_2\}; \tag{1}$$

$$F_1 \cap F_2 \quad \textit{exists in } (\mathscr{F}(\Gamma), \subseteq) \qquad \textit{and} \qquad F_1 \cap F_2 = \inf\{F_1, F_2\}. \tag{2}$$

Proof. By (2.iv) $\sup\{F_1, F_2\}$ and $\inf\{F_1, F_2\}$ are elements of $\mathscr{N}(\Gamma)$, and if $F_1, F_2 \in \mathscr{F}(\Gamma)$ then $\inf\{F_1, F_2\}$ and $\sup\{F_1, F_2\}$ are also elements of $\mathscr{F}(\Gamma)$. $(\sup\{F_1, F_2)(x) \in [0, 1]$ for every $x \in \Gamma$).

On the other hand, $\sup\{F_1, F_2\}$ is an upper bound of $\{F_1, F_2\}$ since $F_1, F_2 \subseteq \sup\{F_1, F_2\}$. Furthermore, if $\hat{F}$ is any fuzzy event which is an upper bound of $\{F_1, F_2\}$ then

$$F_1(x) \leqslant \hat{F}(x) \qquad \text{for every } x \in \Gamma,$$

$$F_2(x) \leqslant \hat{F}(x) \qquad \text{for every } x \in \Gamma,$$

and hence $\sup\{F_1, F_2\}(x) \leqslant \hat{F}(x)$ for every $x \in \Gamma$, i.e., $\sup\{F_1, F_2\} \subseteq \hat{F}$, which implies $F_1 \cup F_2 = \sup\{F_1, F_2\}$. Analogously, it is easy to prove that $F_1 \cap F_2 = \inf\{F_1, F_2\}$.

Generalizing the procedure of the previous proposition, we can also prove that

(σ.1) $\cup\{F_n: n \in \mathbb{N}\}$ exists and $\cup\{F_n: n \in \mathbb{N}\} = \sup\{F_n: n \in \mathbb{N}\}$

(σ.2) $\cap\{F_n: n \in \mathbb{N}\}$ exists and $\cap\{F_n: n \in \mathbb{N}\} = \inf\{F_n: n \in \mathbb{N}\}$.

The σ-lattice $(\mathscr{F}(\Gamma), \subseteq)$ is distributive.

Indeed, it is easy to prove the following properties:

$$F_1 \cup (F_2 \cap F_3) = (F_1 \cup F_2) \cap (F_1 \cup F_3) \qquad \text{for every } F_1, F_2, F_3, \tag{3}$$

$$F_1 \cap (F_2 \cup F_3) = (F_1 \cap F_2) \cup (F_1 \cap F_3) \qquad \text{for every } F_1, F_2, F_3. \tag{4}$$

PROPOSITION 4.2. *The mapping $\mathscr{F}(\Gamma) \to \mathscr{F}(\Gamma)$, $F \to F' = \mathbf{1} - F$ is a strong degenerate orthogomplementation for the poset $(\mathscr{F}(\Gamma), \subseteq, \mathbf{0}, \mathbf{1})$, i.e., it satisfies the conditions:*

$$F'' = F, \tag{5}$$

$$F_1 \subseteq F_2 \quad \textit{implies} \quad F_2' \subseteq F_1'. \tag{6}$$

Proof. Since $\mathbf{1} \in \mathscr{F}(\Gamma)$, the condition (2.iii) ensures that for every $F \in \mathscr{F}(\Gamma)$ the mapping $(\mathbf{1} - F)$ also is an element of $\mathscr{N}(\Gamma)$. Moreover, it is $\sigma(\mathbf{1} - F) \subseteq [0, 1]$ and then $(\mathbf{1} - F) \in \mathscr{F}(\Gamma)$. Obviously $F'' = F$, and let $0 \leqslant F_1(x) \leqslant F_2(x) \leqslant 1$ for every $x \in \Gamma$; then we get $0 \leqslant 1 - F_2(x) \leqslant 1 - F_1(x)$ for every $x \in \Gamma$.

PROPOSITION 4.3. *$F \cup F' = \mathbf{1}$ iff $\sigma(F) \subseteq \{0, 1\}$ iff $F \in \mathscr{E}(\Gamma)$.*

Proof. If $F \cup F' = \mathbf{1}$ then $\sup\{F(x); 1 - F(x)\} = 1$ for every $x \in \Gamma$ and then either $F(x) = 1$ or $1 - F(x) = 1$ for every $x \in \Gamma$, i.e., either $F(x) = 1$ or $F(x) = 0$.

According to [1], the degeneration property of an orthocomplementation satisfying conditions (5) and (6) consists in the fact that the *orthocomplementation Kernel* $\mathscr{N}(') := \{F \in \mathscr{F}(\Gamma): F \subseteq F'\}$ does not coincide with the trivial set $\{\mathbf{0}\}$. Concerning this degeneration, we shall now prove

PROPOSITION 4.4. *In a poset $(\mathscr{F}(\Gamma), \subseteq, \mathbf{0}, \mathbf{1}, ')$ with degenerate orthocomplementation, the following conditions are mutually equivalent:*

(do.i) $\mathscr{N}(') = \{\mathbf{0}\}$;

(do.ii) $F \cap F' = \mathbf{0}$ *for every* $F \in \mathscr{F}(\Gamma)$;

(do.iii) $F \cup F' = \mathbf{1}$ *for every* $F \in \mathscr{F}(\Gamma)$.

Proof. Indeed let $F_0 \subseteq F$ and $F_0 \subseteq F'$, then from this second relation it follows that $F'' = F \subseteq F_0'$. This result and $F_0 \subseteq F$ get $F_0 \subseteq F_0'$. Therefore, if F_0 is a lower bound of $\{F, F'\}$ then $F_0 \in \mathscr{N}(')$ and if (do.i) is true then $F_0 = \mathbf{0}$, concluding that $F \cap F' = \mathbf{0}$ for every $F \in \mathscr{F}(\Gamma)$. Assuming now the validity of (do.ii), let $F_0 \supseteq F'$ and $F_0 \supseteq F$; then by (5) and (6), $F_0' \subseteq F$ and $F_0' \subseteq F'$, and thus $F_0' = \mathbf{0}$ is a consequence of (do.ii). In this case we get $F_0 = \mathbf{0}' = \mathbf{1}$. That is, the unique upper bound of $\{F, F'\}$ is $\mathbf{1}$, i.e., $F \cup F' = \mathbf{1}$. Al last, if $F_0 \in \mathscr{N}(')$, i.e., $F_0 \subseteq F_0'$, then trivially $F_0 \cup F_0' = F_0'$ and if (do.iii) is true, from this result we get $F_0' = \mathbf{1}$ from which $F_0 = \mathbf{1}' = \mathbf{0}$ follows, that is, $\mathscr{N}(') = \{\mathbf{0}\}$.

Concerning the orthocomplementation Kernel we get the following result:

PROPOSITION 4.5. $\mathscr{N}(') := \{F \in \mathscr{F}(\Gamma): \sigma(\Gamma) \subseteq [0, \frac{1}{2}]\}$.

Proof. Let $F \in \mathscr{N}(')$; then $F \subseteq F'$ implies $F(x) \leqslant 1 - F(x)$, $\forall x$ from which $0 \leqslant 2F(x) \leqslant 1$ follows, and then $0 \leqslant F(x) \leqslant \frac{1}{2}$. On the contrary, if $F \in \mathscr{F}(\Gamma)$ is such that $0 \leqslant \mathrm{cl}(\mathrm{range}(F)) \leqslant \frac{1}{2}$ then we have $0 \leqslant F(x) \leqslant \frac{1}{2}$, $\forall x$ from which we get $0 \leqslant F(x) + F(x) \leqslant 1$, i.e., $0 \leqslant F(x) \leqslant 1 - F(x)$, $\forall x$.

COROLLARY. $\mathscr{N}(') = \{\mathbf{0}\}$ *iff* $\mathscr{F}(\Gamma) = \mathscr{E}(\Gamma)$.

Proof. $\mathscr{N}(') = \{\mathbf{0}\}$ implies $F \cup F' = \mathbf{1}$ for every F and then, from Proposition 4.3, we get $F \in \mathscr{E}(\Gamma)$.

Remark. The equivalence between the propositions "$\mathscr{N}(') = \{\mathbf{0}\}$" and "$F \cup F' = \mathbf{1}$ for every $F \in \mathscr{F}(\Gamma)$" implies that the orthocomplementation introduced in Proposition 4.2 is not degenerate iff $\mathscr{F}(\Gamma) = \mathscr{E}(\Gamma)$.

From this degenerate orthocomplementation it is possible to induce a degenerate orthogonality relation

$$F_1 \perp F_2 \quad \text{iff} \quad F_1 \subseteq F_2', \tag{7}$$

which is an extension to $\mathscr{F}(\Gamma)$ of the orthogonality relation defined in Section 2 by (14.a); indeed,

$$F_1 \perp F_2 \quad \text{iff} \quad 0 \leqslant F_1(x) + F_2(x) \leqslant 1 \qquad \text{for every } x \in \Gamma. \tag{8}$$

Also, properties (15), (17), Section 2, are extended in the following way

$$F_1 + F_2 \in \mathscr{F}(\Gamma) \quad \text{iff} \quad F_1 \perp F_2, \tag{9}$$

and in this case we shall often write $F \dotplus F$.

$$F_1 \cdot F_2 \in \mathscr{F}(\Gamma) \qquad \text{for every } F_1, F_2 \in \mathscr{F}(\Gamma), \tag{10}$$

$$\lambda \in [0, 1] \quad \text{and} \quad F \in \mathscr{F}(\Gamma) \quad \text{imply} \quad \lambda F \in \mathscr{F}(\Gamma). \tag{11}$$

In the following $\{F_n: n \in \mathbb{N}\} \perp_{\mathrm{tot}}$ stands for $0 \leqslant \sum_{n \in \mathbb{N}} F_n(x) \leqslant 1$ for every $x \in \Gamma$ while $\{F_n: n \in \mathbb{N}\} \perp$ will denote the fact that $F_i \perp F_1$ for every $i \neq j$.

Of course

$$\{F_n\colon n\in\mathbb{N}\}\ \perp_{\text{tot}} \qquad \text{iff} \qquad \sum_{n\in\mathbb{N}} F_n \in \mathscr{F}(\Gamma), \tag{12}$$

$$\{F_n\colon n\in\mathbb{N}\}\ \perp_{\text{tot}} \qquad \text{implies} \qquad \{F_n\colon n\in\mathbb{N}\}\ \perp. \tag{13}$$

On $\mathscr{F}(\Gamma)$ we can introduce other operations. In the first place, we define the *difference* $F_1\backslash F_2$ of two fuzzy events as

$$F_1\backslash F_2 := F_1\cdot F_2' = F_1\cdot(1-F_2), \tag{14}$$

which still is a fuzzy event. This last result assures us that the *sum*

$$F_1\oplus F_2 := F_1+F_2-F_1\cdot F_2 \tag{15}$$

of two fuzzy events is a fuzzy event.

Indeed, $F_2\leqslant 1$ implies $F_2\cdot F_1'\leqslant F_1' = 1-F_1$ and then

$$F_1+F_2\cdot F_1' = F_1+F_2-F_1\cdot F_2 = F_1\oplus F_2\leqslant 1.$$

From (15) it follows that

$$\begin{aligned} F_1\oplus F_2 &= F_1\backslash F_2 \dotplus F_2\backslash F_1 \dotplus F_1\cdot F_2 \\ &= F_1 \dotplus F_2\backslash F_1 \\ &= F_2 \dotplus F_1\backslash F_2 \end{aligned} \tag{16}$$

and then

$$\{F_1\backslash F_2, F_2\backslash F_1, F_1\cdot F_2\}\ \perp_{\text{tot}}. \tag{17}$$

In the case of exact events E_1 and E_2, the relation (16) expresses the trivial property $E_1\cup E_2 = (E_1\cup E_2{}^C)\cup(E_2\cap E_1{}^C)\cup(E_1\cap E_2)$.

Moreover, there are some interesting relations between the ordered structures of fuzzy and exact events with regard to the mappings introduced in Section 3:

$$E_m\colon \mathscr{F}(\Gamma)\to\mathscr{E}(\Gamma)$$
$$E_M\colon \mathscr{F}(\Gamma)\to\mathscr{E}(\Gamma),$$

which assign to each fuzzy event the corresponding least and greatest exact event, respectively.

In the first place, E_m and E_M are order preserving because they satisfy

$$F_1\subseteq F_2 \qquad \text{implies} \qquad E_m(F_1)\subseteq E_m(F_2); \tag{18}$$

$$F_1\subseteq F_2 \qquad \text{implies} \qquad E_M(F_1)\subseteq E_M(F_2). \tag{19}$$

Relative to the σ-meet and the σ-join operations defined by (σ.1) and (σ.2) we get that E_m and E_M are morphisms of σ-lattices, i.e.,

$$E_m(\cap F_j) = \cap E_m(F_j), \tag{20}$$

$$E_M(\cap F_j) = \cap E_M(F_j), \tag{21}$$

$$E_m(\cup F_j) = \cup E_m(F_j), \tag{22}$$

$$E_M(\cup F_j) = \cup E_M(F_j). \tag{23}$$

These properties are an immediate consequence of the following obvious relations:

$$\{x \in \Gamma : \inf\{F_j(x)\} = 1\}\} = \cap\{x : F_j(x) = 1\}$$
$$\{x \in \Gamma : \sup\{F_j(x)\} = 1\}\} = \cup\{x : F_j(x) = 1\}$$

and

$$\{x \in \Gamma : \inf\{F_j(x)\} \in (0, 1]\} = \cap\{x : F_j(x) \in (0, 1]\}$$
$$\{x \in \Gamma : \sup\{F_j(x)\} \in (0, 1]\} = \cup\{x : F_j(x) \in (0, 1]\}.$$

On the contrary, the links between the corresponding orthocomplementations of $\mathscr{F}(\Gamma)$ and $\mathscr{E}(\Gamma)$ do not present the same regular feature, since

$$E_m(F') = [E_M(F)]^C, \tag{24}$$

$$E_M(F') = [E_m(F)]^C. \tag{25}$$

5. Classical Fuzzy Logic of Weak Type and Partial Baer*-Ring

The poset $\mathscr{F}(\Gamma)$ can be equipped with another orthogonality relation which generalized condition (14b), Section 2, which is equivalent to the canonical orthogonality on $\mathscr{E}(\Gamma)$. This orthogonality is weak (and nondegenerate) and can be induced by a weak (and nondegenerate) orthocomplementation.

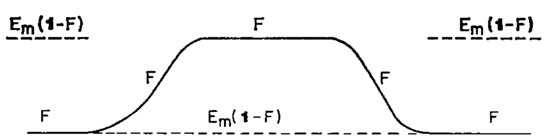

Figure 2

Let $F \in \mathscr{F}(\Gamma)$; then $(F^{-1}(0, 1])^C \in \mathscr{E}(\Gamma)$ by the measurability of F. The element of $\mathscr{E}(\Gamma)$ defined by

$$\chi_{(F^{-1}(0,1])^C}(x) = 1, \qquad F(x) = 0$$
$$= 0, \qquad F(x) \neq 0$$

is the least exact event associated to the fuzzy event $F' = \mathbf{1} - F$, that is $\chi_{(F^{-1}(0,1])^C} = E_m(\mathbf{1} - F)$.

PROPOSITION 5.1. *The mapping*

$$\mathscr{F}(\Gamma) \to \mathscr{E}(\Gamma), \qquad F \to F^\wedge = \chi_{(F^{-1}(0,1])^C}$$

defines a weak nondegenerate orthocomplementation, i.e., satisfies the conditions:

(i) $F \subseteq F^{\wedge\wedge}$;

(ii) $F_1 \subseteq F_2$ *implies* $F_2^\wedge \subseteq F_1^\wedge$;

(iii) $F \subseteq F^\wedge$ *implies* $F = \mathbf{0}$.

Proof. In fact

$$F^{\wedge\wedge} = \chi_{(F^{-1}(0,1])} = E_M(F)$$

is the greatest exact event associated to F.

Let $F_1(x) \leqslant F_2(x)$; then $F_1^{-1}(0, 1] \subseteq F_2^{-1}(0, 1]$ from which $(F_2^{-1}(0, 1])^C \subseteq (F_1^{-1}(0, 1])^C$ follows, and then $F_2^\wedge(x) \leqslant F_1^\wedge(x)$. Moreover, if

$$0 \leqslant F(x) \leqslant 1, \qquad F(x) = 0$$
$$\leqslant 0, \qquad F(x) \neq 0$$

then $F(x) = 0$ for every x.

The orthogonality (of course, weak and nondegenerate) induced by the orthocomplementation now introduced is defined by

$$F_1 \underline{\perp} F_2 \qquad \text{iff} \qquad F_1 \subseteq F_2^\wedge,$$

and the following proposition shows that $\underline{\perp}$ is a generalization of (14b), Section 2:

PROPOSITION 5.2. *The following conditions are equivalent:*

(i) $F_1 \subseteq F_2^\wedge$,

(ii) $F_1 \cap F_2 = \mathbf{0}$ $(\inf\{F_1, F_2\} = \mathbf{0})$,

(iii) $F_1 \cdot F_2 = \mathbf{0}$,

(iv) $\operatorname{supp}(F_1) \cap \operatorname{supp}(F_2) = \varnothing$.

Proof. $F_1 \subseteq F_2^\wedge \Rightarrow F_1(x) \leqslant \chi_{(F^{-1}(0,1])}(x) \Rightarrow \inf\{F_1, F_2\} = \mathbf{0} \Rightarrow F_1 \cdot F_2 = \mathbf{0} \Rightarrow \operatorname{supp}(F_1) \cap \operatorname{supp}(F_2) = \varnothing \Rightarrow F_1 \subseteq F_2^\wedge$.

We remember that a nondegenerate weak orthogonality relation satisfies the conditions

$$F \cap F^\wedge = \mathbf{0} \qquad \text{for every } F \in \mathscr{F}(\Gamma),$$

but in general, $F \cup F^{\wedge} = \sup\{F, E_m(1 - F)\}$ is different from $\mathbf{1}$ and

$$F \cup F^{\wedge} = \mathbf{1} \qquad \text{iff} \qquad F \in \mathscr{E}(\Gamma). \tag{1}$$

Moreover,

$$F_1 \perp F_2 \qquad \textit{implies} \qquad F_1 \oplus F_2 = F_1 \dotplus F_2 = F_1 \cup F_2 \quad \textit{and} \quad F_1 \backslash F_2 = F_1. \tag{2}$$

We shall observe that $F^{\wedge} = E_m(1 - F)$ is a Brouwerian complement and, more generally, that $\mathscr{F}(\Gamma)$ is a Brouwerian σ-lattice. Exactly, if $G \in \mathscr{F}(\Gamma)$, the least G-event assigned to each $F \in \mathscr{F}(\Gamma)$ is defined by

$$\begin{aligned}[E_G(F)](x) &:= 1, & 1 \leqslant G(x) + F(x) \\ &:= G(x), & F(x) + G(x) < 1,\end{aligned}$$

and the Brouwerian relative complement of F in G is then $E_G(\mathbf{1} - F)$. Therefore $E_m(\mathbf{1} - F) = E_0(\mathbf{1} - F)$.

For a more detailed discussion about this argument see De Luca and Termini [3]; we only remark that from this point of view $\mathscr{F}(\Gamma)$ can be regarded as a Brouwerian logic.

Now let $\{F_n: n \in \mathbb{N}\} \perp$ stand for $F_i \perp F_j$ for $i \neq j$. We easily deduce from (iv), Proposition 5.2, that

$$\{F_n: n \in \mathbb{N}\} \perp \qquad \text{implies} \qquad \{F_n: n \in \mathbb{N}\} \perp_{\text{tot}}$$

and

$$\sum F_n = \bigcup F_n = \oplus F_n.$$

This last result gives us a relation between the degenerate orthogonality (6) and the weak orthogonality. Moreover, we can conclude that ($\mathscr{F}(\Gamma)$, $\subseteq$, $\mathbf{0}$, $\mathbf{1}$, $\cdot^{\wedge}$) *is a weak orthocomplemented* $\sigma\sum$*-orthocomplete* σ*-lattice*, *i.e.*, a σ-lattice with weak orthocomplementation such that for every sequence $\{F_n: n \in \mathbb{N}\} \perp$ of mutually $\perp$-orthogonal elements we get that $\sum F_n$ is an element of $\mathscr{F}(\Gamma)$.

However, from the point of view of the analogies with the mathematical structure of the axiomatic foundation of quantum mechanics in Hilbert spaces, we are more interested in connecting this weak orthogonality to the algebraic operations on fuzzy events. In this manner, the notable property that $\mathscr{F}(\Gamma)$ is closed with regard to product, is entirely utilized.

Precisely, let us consider

$$\hat{\mathscr{F}}(\Gamma) := \{\hat{F}: \Gamma \to \mathbb{C} \mid \hat{F} \in \mathscr{N}(\Gamma), |\hat{F}| \in \mathscr{F}(\Gamma)\},$$

the set of measurable elements $\hat{F}$ of $\mathscr{N}(\Gamma)$ such that $\sigma(\hat{F})$ is contained in the closed unit circle of the complex plane. These elements are called *complex fuzzy events*.

The set $\hat{\mathscr{F}}(\Gamma)$ can be considered as an abelian Baer*-semigroup with zero and unity:

$$(\hat{\mathscr{F}}(\Gamma), \circ, \mathbf{0}, \mathbf{1}, {}^*, {}'),$$

where $\circ: \hat{\mathscr{F}}(\Gamma) \times \hat{\mathscr{F}}(\Gamma) \to \hat{\mathscr{F}}(\Gamma)$ is the usual product between functions, observing that this definition is well defined by the property $|\hat{F}_1 \circ \hat{F}_2| = |\hat{F}_1| \cdot |\hat{F}_2|$. The structure $(\hat{\mathscr{F}}(\Gamma), \bigcirc)$ is obviously an abelian semigroup whose *unity* is the fuzzy event $\mathbf{1} \in \hat{\mathscr{F}}(\Gamma)$ and whose *zero* is the fuzzy event $\mathbf{0} \in \hat{\mathscr{F}}(\Gamma)$.

The mapping $^*: \hat{\mathscr{F}}(\Gamma) \to \hat{\mathscr{F}}(\Gamma)$, which assigns to each $\hat{F} \in \hat{\mathscr{F}}(\Gamma)$ the corresponding complex coniugate function $\hat{F}^* \in \hat{\mathscr{F}}(\Gamma)$, is a semigroup involution, i.e., it satisfies the conditions

$$(\hat{F}^*)^* = \hat{F} \qquad \text{for every } \hat{F} \in \hat{\mathscr{F}}(\Gamma), \tag{3}$$

$$(\hat{F}_1 \circ \hat{F}_2)^* = \hat{F}_2^* \circ \hat{F}_1^* \qquad \text{for every } \hat{F}_1, \hat{F}_2 \in \hat{\mathscr{F}}(\Gamma). \tag{4}$$

In this manner $(\hat{\mathscr{F}}(\Gamma), \circ, {}^*)$ is an involution semigroup in which we can select the set $\mathscr{P}(\Gamma)$ of *projections*, where

$$\mathscr{P}(\Gamma) := \{\hat{F} \in \hat{\mathscr{F}}(\Gamma): \hat{F} \circ \hat{F} = \hat{F}^* = \hat{F}\} = \{\chi_E: E \in \mathscr{E}(\Gamma)\} = \Pi(\Gamma).$$

From the theory of Baer*-semigroup (see, for example, [5–7, 14]) we know that the relation $\leqslant$ on $\mathscr{P}(\Gamma)$ is defined as follows:

$$E_1 \leqslant E_2 \quad \text{iff} \quad E_1 = E_1 \circ E_2 \tag{5}$$

is an order relation intrinsic to the structure $(\hat{\mathscr{F}}(\Gamma), \circ, {}^*)$ and in our case it is exactly the canonical order introduced on $\Pi(\Gamma)$ in Section 2 by (12) or (12a).

The set of fuzzy events is exactly the set

$$\mathscr{F}(\Gamma) := \{F \in \hat{\mathscr{F}}(\Gamma): F = |F| = F^*\}$$

and, obviously, it is a subsemigroup of $\hat{\mathscr{F}}(\Gamma)$. Of course, the projections are also fuzzy events.

At last, $': \hat{\mathscr{F}}(\Gamma) \to \mathscr{P}(\Gamma)$, is the mapping defined as follows:

$$\hat{F}' = (|\hat{F}|^{-1}(0, 1])^C. \tag{6}$$

If $\hat{F}_0$ is any fixed element in $\hat{\mathscr{F}}(\Gamma)$ then

$$\{\hat{F} \in \hat{\mathscr{F}}(\Gamma): \hat{F}_0 \circ \hat{F} = \mathbf{0}\} = \{\hat{F} \in \hat{\mathscr{F}}(\Gamma): \hat{F} = \hat{F}_0' \circ \hat{F}\}.$$

With this mapping $\hat{\mathscr{F}}(\Gamma)$ becomes a Baer*-semigroup, whose set of *closed projections*, denoted by $\mathscr{P}'(\Gamma)$ and defined as

$$\mathscr{P}'(\Gamma) := \{E \in \mathscr{P}(\Gamma): (E')' = E\}$$

is the set of exact events $\Pi(\Gamma)$. Therefore, in the Baer*-semigroup $\hat{\mathscr{F}}(\Gamma)$ the set of projections is identical to the set of closed projections:

$$\mathscr{P}(\Gamma) = \mathscr{P}'(\Gamma) = \Pi(\Gamma).$$

The restriction of the mapping (6) to $\mathscr{F}(\Gamma)$ is the orthocomplementation introduced by Proposition 4.4, while the restriction of the mapping (6) to $\Pi(\Gamma)$ is the canonical orthocomplementation of $\Pi(\Gamma)$.

The following theorem on Baer*-semigroups [14] explains the behavior of the lattice operations on the Boolean σ-lattice $\mathscr{E}(\Gamma)$ with regard to the product and orthocomplementation operations.

THEOREM 5.3. *Let* $(\hat{\mathscr{F}}(\Gamma), \circ, 0, 1, {}^*, {}')$ *be an abelian Baer*-semigroup, and let* $(\mathscr{E}(\Gamma), \leqslant, 0, 1, {}')$ *be the subset of closed projections with order* (5) *and orthocomplementation* (6); *then* $\mathscr{E}(\Gamma)$ *is a Boolean σ-lattice such that*

(a) $E_1 \wedge E_2 = (E_1 \circ E_2)' \circ E_2 = E_1 \circ E_2$.

(b) If E_1, $E_1 \in \mathscr{E}(\Gamma)$, then the following propositions are equivalent:

(i) there exists $\{\tilde{E}_1, \tilde{E}_2, E_{12}\} \subseteq \mathscr{E}(\Gamma)$ such that

$$\{\tilde{E}_1, \tilde{E}_2, E_{12}\} \perp$$

and

$$E_1 = \tilde{E}_1 \cup E_{12}, \qquad E_2 = \tilde{E}_2 \cup E_{12};$$

(ii) $E_1 \circ E_2 = E_2 \circ E_1$.

In the generalized approach to axiomatic foundations of quantum mechanics (see, for instance, Varadarajan [20, 21], Gudder [10–12]), the property (b.i) is called condition of *compatibility* of the two events E_1 and E_2, written $E_1 \leftrightarrow E_2$, and is introduced also on an orthocomplemented poset $(\mathscr{E}, \leqslant, 0, 1, {}')$, which satisfies weaker conditions, i.e., the conditions of orthomodular orthoposet.

In the generalized case, it might happen that two exact events are not compatible but the set of exact events of a classical information system has the property that every couple of exact events is compatible.

We shall observe that relative to the mappings E_m and E_M introduced in Section 3, if F_1 and F_2 belong to $\mathscr{F}(\Gamma)$ then

$$E_m(F_1 \circ F_2) = E_m(F_1 \cap F_2), \tag{7}$$

$$E_M(F_1 \circ F_2) \subseteq E_M(F_1 \cap F_2). \tag{8}$$

Besides the structure of Baer*-semigroup, which involves the product operation defined on complex fuzzy events, we can utilize the weak orthogonality relation $\underline{\perp}$ to introduce on $\hat{\mathscr{F}}(\Gamma)$ an operation of partial sum. Extending the weak orthogonality introduced on $\mathscr{F}(\Gamma)$ by Proposition 5.2 we can set

$$\hat{F}_1 \underline{\perp} \hat{F}_2 \qquad \text{iff} \qquad \hat{F}_1 \circ \hat{F}_2 = \mathbf{0} \qquad (\text{i.e., } \inf\{\hat{F}_1, \hat{F}_2\} = \mathbf{0}),$$

which, according to the Randall definition (see [8, 16]), is an orthogonality on a set without order. In this case, denoted by $(\hat{\mathscr{F}}(\Gamma) \times \hat{\mathscr{F}}(\Gamma))_{\underline{\perp}} = \{(\hat{F}_1, \hat{F}_2) \in \hat{\mathscr{F}}(\Gamma) \times \hat{\mathscr{F}}(\Gamma) : \hat{F}_1 \underline{\perp} \hat{F}_2\}$ we have that the partial operation of sum

$$+ : (\hat{\mathscr{F}}(\Gamma) \times \hat{\mathscr{F}}(\Gamma))_{\underline{\perp}} \to \hat{\mathscr{F}}(\Gamma)$$

defined by

$$(\hat{F}_1 + \hat{F}_2)(x) := \hat{F}_1(x) + \hat{F}_2(x)$$

is well defined, and we can consider the triple

$$(\hat{\mathscr{F}}(\Gamma), \underline{\perp}, +),$$

which is an abelian partial group; i.e.,

$$\{\hat{F}_1, \hat{F}_2\} \underline{\perp} \qquad \text{implies} \qquad \hat{F}_1 + \hat{F}_2 = \hat{F}_2 + \hat{F}_1; \tag{9}$$

$$\begin{gathered} \text{there exists } \mathbf{0} \in \hat{\mathscr{F}}(\Gamma) \text{ such that} \\ \{\mathbf{0}, \hat{F}\} \underline{\perp} \qquad \text{for every } \hat{F} \in \hat{\mathscr{F}}(\Gamma) \quad \text{and} \quad \mathbf{0} + \hat{F} = \hat{F} + \mathbf{0} = \hat{F}, \end{gathered} \tag{10}$$

$$\begin{gathered} \text{for every } \hat{F} \in \hat{\mathscr{F}}(\Gamma), \text{ there exists } -\hat{F} \in \hat{\mathscr{F}}(\Gamma) \text{ such that} \\ \hat{F} + (-\hat{F}) = (-\hat{F}) + \hat{F} = \mathbf{0} \qquad (\text{note that } \{\hat{F}, -\hat{F}\} \underline{\perp} \text{ iff } \hat{F} = \mathbf{0}), \end{gathered} \tag{11}$$

let $\{\hat{F}_1, \hat{F}_2, \hat{F}_3\} \underline{\perp}$; then

$$\hat{F}_1 + (\hat{F}_2 + \hat{F}_3) = (\hat{F}_1 + \hat{F}_2) + \hat{F}_3. \tag{12}$$

Moreover, the distributivity law hold:

$$\{\hat{F}_1, \hat{F}_2\} \underline{\perp} \qquad \text{implies} \qquad \hat{F} \circ (\hat{F}_1 + \hat{F}_2) = \hat{F} \circ \hat{F}_1 + \hat{F} \circ \hat{F}_2. \tag{13}$$

In this way $(\hat{\mathscr{F}}(\Gamma), \underline{\perp}, +, \circ, {}^*, {}')$ is a partial Baer*-ring and its positive self-adjoint part $(\mathscr{F}(\Gamma), \underline{\perp}, +, \circ, {}')$ is closed relative to the sum operation now operation but in this case it is not true that $F \in \mathscr{F}(\Gamma)$, with $F \neq \mathbf{0}$, implies $-F \in \mathscr{F}(\Gamma)$, i.e., $\mathscr{F}(\Gamma)$ is an abelian partial semigroup with unit as regards to sum operations.

The behavior of the mappings E_m and E_M as to the partial sum operation is given by the following properties:

$$E_m(F_1 + F_2) \supseteq E_m(F_1 \cup F_2), \tag{14}$$

$$E_M(F_1 + F_2) = E_M(F_1 \cup F_2). \tag{15}$$

At last we can consider the mapping

$$|\ |: \hat{\mathscr{F}}(\Gamma) \to \mathscr{F}(\Gamma), \qquad \hat{F} \to |\hat{F}|$$

for which we get that

$$|(|\hat{F}|)| = |\hat{F}| \qquad \text{for every } \hat{F} \in \hat{\mathscr{F}}(\Gamma); \tag{16}$$

$$|\hat{F}_1 \circ \hat{F}_2| = |\hat{F}_1| \circ |\hat{F}_2| \qquad \text{for every } \hat{F}_1, \hat{F}_2 \in \hat{\mathscr{F}}(\Gamma); \tag{17}$$

$$|\hat{F}_1 + \hat{F}_2| \leqslant |\hat{F}_1| + |\hat{F}_2| \qquad \text{for every } \{F_1, F_2\}_\perp . \tag{18}$$

This mapping sends the set of complex fuzzy events into the set of fuzzy events and satisfies in particular the property of idempotence. For this reason we shall say that $|\ |$ is the canonical *fuzzy projection.*

6. Classical Fuzzy Logic of Emperical Type

The classical fuzzy logics previously studied, both the degenerate and the weak ones, are relative to the order relation introduced by (6), Section 3. With this order, *the set of fuzzy events is a distributive σ-lattice embedded into the partial Baer*-ring of complex fuzzy events.* This order is, in a certain sense, "intrinsic" to the concrete structure of $\mathscr{F}(\Gamma)$, as functional space, rather then to the "physical" structure of the classical information system. Since it is this last that can be generalized owing to the analogies with similar concrete structures which arise in other physical situations, such as the Hilbert-space model of quantum mechanics, we are more interested to the empirical order defined by (5), Section 3. We shall called this order "empirical" for if F_1 and F_2 are two fuzzy events we have that F_2 is "empirically" greater or equal to F_1, inside the considered information system, if and only if for every preparation procedure available in our information system, we get that the probability of occurence of F_2 is always greater or equal to the probability of occurence of F_1.

Property (7), Section 3, gives us the link between the intrinsic ($\subseteq$) and the empirical ($\leqslant$) logics. Denoted by $F_1 \vee F_2$ and $F_1 \wedge F_2$, respectively, the lub and the glb of $\{F_1, F_2\}$ relative to the empirical order, the most relevant feature of the empirical logic compared with the intrinsic one is that, while $(\mathscr{F}(L), \subseteq)$ is a σ-lattice, (and then in particular there exist $F_1 \cup F_2$ and $F_1 \cap F_2$) in general we cannot say that the corresponding elements $F_1 \vee F_2$ and $F_1 \wedge F_2$ exist with regard to $(\mathscr{F}(\Gamma), \leqslant)$. This is because, for instance, besides the upper bounds of $\{F_1, F_2\}$ relative to the intrinsic order, which of course are also upperbounds of $\{F_1, F_2\}$ relative to the empirical order, there could be other empirical upper bounds with the possible result that $F_1 \vee F_2$ does not exist.

At any rate, we observe that if $F_1 \vee F_2$ exists then

$$F_1 \vee F_2 \leqslant F_1 \cup F_2 .$$

However, the empirical fuzzy logic is in general a poset with the property that the mapping $F' = 1 - F$ defined by Proposition 4.2 is again a degenerate orthocomplementation. Indeed $F'' = F$ is trivially true for every $F \in \mathscr{F}(\Gamma)$ and moreover, the condition that

$$F_1 \leqslant F_2 \qquad \text{implies} \qquad F_2' \leqslant F_1' \tag{1}$$

follows from $0 \leqslant \mu(F) + \mu(F') = \mu(1)$ for every $F \in \mathscr{F}(\Gamma)$ and every $\mu \in \mathscr{B}(\Gamma)$ In general, no other order properties are satisfied by a classical information system. In particular, the element $F \vee F'$ in general does not exist and if $F \vee F'$ exists for a certain $F \in \mathscr{F}(\Gamma)$ then in general $F \vee F' \leqslant 1$.

An extreme case, from this point of view, is that of a *totally* or *atomic* classical information system, i.e., of a classical information system for which the following conditions hold:

(tc.i) $\{x\} \in \mathscr{E}(\Gamma)$ for every $x \in \Gamma$.

(tc.ii) The Dirac measures $\mu_{\{x\},k}$, defined as

$$\begin{aligned} \mu_{\{x\},k}(E) &:= k, & \{x\} \subseteq E \\ &:= 0, & \{x\} \subseteq E^C, \end{aligned}$$

are in $\mathscr{B}(\Gamma)$ for every $\{x\}$ and every $k \in \mathbb{R}_+$.

In this case the intrinsic and the empirical orders are the same:

$$F_1 \subseteq F_2 \qquad \text{iff} \qquad F_1 \leqslant F_2 . \tag{2}$$

Indeed, from (7), Section 3, we must prove only that $F_1 \leqslant F_2$ implies $F_1 \subseteq F_2$. But $F_1 \leqslant F_2$ implies that $\mu(F_1) \leqslant \mu(F_2)$ for every $\mu \in \mathscr{B}(\Gamma)$ and then in particular, setting more somply $\mu_{\{x\}}$ instead of $\mu_{\{x\},1}$, we get that

$$F_1(x) = \mu_{\{x\}}(F_1) \leqslant \mu_{\{x\}}(F_2) = F_2(x)$$

is true for every $x \in \Gamma$, i.e., $F_1 \subseteq F_2$.

Therefore, the empirical logic of fuzzy events in a totally classical information system is also a distributive σ-lattice with degenerate orthocomplementation.

Generalizing the statistical mechanics interpretative rules we can regard Γ as a generalized phase-pace, the elements of Γ as the "*microstates*" of the physical system, and any singleton $\{x\}$ as the event that the microstate of the system is exactly x. By the property that

$$\mu_{\{x\},k}(\{x\}) = \mu_{\{x\},k}(\Gamma),$$
$$\mu_{\{x\},k}(\Gamma \backslash \{x\}) = 0,$$

we get that $\mu_{\{x\},k}$, for any $k \in \mathbb{R}_+$, is a preparation procedure, performed by certain macroscopic apparatuses, which provide ensembles of physical systems with the exact microstate x.

Another interesting case is that of a classical information system whose total structure is such that the induced empirical fuzzy logic is an orthomodular σ-orthoposet. This is the logic of a physical system which exhibits quantum effects according to the generalized approach to axiomatic foundation of quantum mechanics (see, for instance, [11, 14, 20]). Unlike the totally classical information systems, the quantum logics require at least that

$$F \vee F' = 1 \qquad \text{for every } F \in \mathscr{F}(\Gamma), \tag{3}$$

that is, the *orthocomplementation must be not degenerate* relative to the empirical order, and that

$$\text{if } \{F_n\colon n \in \mathbb{N}\} \perp \text{ then } \vee F_n \text{ exists in } \mathscr{F}(\Gamma)\text{; that is, } (\mathscr{F}(\Gamma), \leqslant, ') \tag{4}$$

must be a *σ-orthoposet.*

In this case $(\mathscr{F}(\Gamma), \mathscr{B}(\Gamma))$ is a *quantum information system* for which $(\Gamma, \mathscr{E}(\Gamma), \mathscr{B}(\Gamma))$ can be regarded as the underlying *classical pattern.* We shall observe that if all the Dirac measures belong to $\mathscr{B}(\Gamma)$, then $F \cup F' = 1$ for every $F \in \mathscr{F}(\Gamma)$ follows from condition (2), and thus from Proposition 4.3 we get that $\mathscr{F}(\Gamma) = \mathscr{E}(\Gamma)$.

In this way the corresponding quantum logic is the classical one and it does not turn out to be very interesting from the point of view of considering quantum systems as fuzzy structures of a classical information pattern. Therefore, if we want a quantum information system which is not trivially coinciding with its classical pattern, some classes $\{\mu_{\{x\},k}\colon k \in \mathbb{R}_+\}$ of Dirac measures, for a fixed $x \in \Gamma$, must not belong to $\mathscr{B}(\Gamma)$. Physically, this means that it is not possible with the experimental apparatuses available in $\mathscr{B}(\Gamma)$ to prepare the physical system in all the possible microstates of Γ. In a certain sense, this is a weak form of the indetermination principle for which some microstates can be prepared only in a fuzzy way.

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